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NP-Hardness of Linear Multiplicative Programming and Related Problems

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Abstract. The linear multiplicative programming problem minimizes a product of two (positive) variables subject to linear inequality constraints. In this paper, we show NP-hardness of linear multiplicative programming problems and related problems.

Key words: NP-hard, minimization of products, linear multiplicative programming, linear fractional programming, multi-ratio programming.

1. Introduction

In this note, we consider the following problems:

(P1)	(P2)	(P3)
minimize $x_1 x_2$	minimize $x_1 - 1/x_2$	maximize $1/x_1 + 1/x_2$
subject to $A\boldsymbol{x} \leq \boldsymbol{b}$,	subject to $A\boldsymbol{x} \leq \boldsymbol{b}$,	subject to $A\boldsymbol{x} \leq \boldsymbol{b}$,

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$ is a d-dimensional real-valued vector and the feasible region $\Omega = \{\boldsymbol{x} \in \mathcal{R}^d \mid A\boldsymbol{x} \leq \boldsymbol{b}\}$ satisfies the condition that for any feasible vector $\boldsymbol{x}' \in \Omega$, $x'_1, x'_2 > 0$. Problem P1 is called a linear multiplicative programming problem. The above problems arise in many application settings, see the survey [8] and the forthcoming book [9]. For solving the above problems, there exist many algorithms [1,4-7,10,11,13-15]. In the recent paper [12], Pardalos and Vavasis asked the question whether linear multiplicative programming problems are polynomially solvable or not. The purpose of this paper is to show NP-hardness of Problems P1, P2 and P3.

In [12], Pardalos and Vavasis proved that the following quadratic concave optimization problem is NP-hard:

$$\text{(P4)} \quad \text{minimize } x_1 - x_2^2 \quad \text{subject to } A\boldsymbol{x} \leq \boldsymbol{b}.$$

We will begin the next section by refining on the proof of NP-hardness of P4 described in [12].Our new proof offers the key to main results.

2. Preliminaries

As a beginning, we will examine how to calculate the square of a number. Given a vector $\boldsymbol{x} \in [0,1]^n$ and a positive integer number p, the value $px_1 + p^2x_2 + p^3x_3 + \cdots + p^n x_n$ is denoted by $[\boldsymbol{x}]_p$. For any vector $\boldsymbol{x} \in [0,1]^n$, the square of $[\boldsymbol{x}]_p$ is obtained by the equation:

$$([\boldsymbol{x}]_p)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} p^{i+j} x_i x_j.$$

Now, we describe a method to approximate $([\boldsymbol{x}]_p)^2$ by a linear inequality system. When $i \neq j$, we replace the term $x_i x_j$ by a variable y_{ij} satisfying linear inequalities:

$$0 \le y_{ij} \le 1, \quad y_{ij} \le x_i, \quad y_{ij} \le x_j, \quad y_{ij} \ge x_i + x_j - 1. \tag{1}$$

For all i, we replace $x_i x_i$ by a variable y_{ii} satisfying:

$$y_{ii} = x_i. \tag{2}$$

By using y variables, the square of $[\boldsymbol{x}]_p$ is approximated by:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} p^{i+j} y_{ij}.$$

Linear inequalities (1) imply that if either x_i or x_j is 0–1 valued, then $y_{ij} = x_i x_j$. The equality (2) implies that $x_i \in [0,1]$ is 0–1 valued if and only if $y_{ii} = x_i x_i$. So, for any 0–1 valued vector $\boldsymbol{x}$, the equality $([\boldsymbol{x}]_p)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} p^{i+j} y_{ij}$ holds. However, if a given vector $\boldsymbol{x} \in [0,1]^n$ is not 0–1 valued, the equality does not hold in general. Now we consider the difference between $([\boldsymbol{x}]_p)^2$ and $\sum_{i=1}^{n}\sum_{j=1}^{n} p^{i+j} y_{ij}$, when $\boldsymbol{x}$ is not 0–1 valued.

THEOREM 2.1. *Let $\boldsymbol{x} \in \mathcal{R}^n$ and $\boldsymbol{y} \in \mathcal{R}^{n\times n}$ be a pair of vectors satisfying:*

$$\begin{array}{ll} 0 \le x_i \le 1 & (\textit{for all } i), \\ 0 \le y_{ij} \le 1 & (\textit{for all } i,j), \\ y_{ij} \le x_i,\ y_{ij} \le x_j,\ y_{ij} \ge x_i + x_j - 1 & (\textit{for all } i,j \textit{ such that } i \neq j), \\ y_{ii} = x_i & (\textit{for all } i). \end{array} \tag{3}$$

Assume that p is a positive integer, $\boldsymbol{x}$ is not 0–1 valued and there exists a positive value $0 < \varepsilon < 1/2$ satisfying that each element x_i is either $x_i = 0, 1$ or $\varepsilon < x_i < 1 - \varepsilon$. Then the inequality $\sum_{i=1}^{n}\sum_{j=1}^{n} p^{i+j} y_{ij} - ([\boldsymbol{x}]_p)^2 > p^2\varepsilon/2 - pn^2$ holds.

Proof. Let k be the largest index satisfying $0 < x_k < 1$. For any index $i > k$, x_i is 0–1 valued and so $y_{ij} = x_i x_j$ for all j. Then we have the following inequalities;

$$\sum_{i=1}^{n}\sum_{j=1}^{n} p^{i+j} y_{ij} - ([\boldsymbol{x}]_p)^2$$

$$
\begin{aligned}
&= \sum_{i=1}^{k}\sum_{j=1}^{k} p^{i+j} y_{ij} + \sum_{i=k+1}^{n}\sum_{j=1}^{k} p^{i+j} y_{ij} \\
&\quad + \sum_{i=1}^{k}\sum_{j=k+1}^{n} p^{i+j} y_{ij} + \sum_{i=k+1}^{n}\sum_{j=k+1}^{n} p^{i+j} y_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} p^{i+j} x_i x_j \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} p^{i+j} y_{ij} + \sum_{i=k+1}^{n}\sum_{j=1}^{k} p^{i+j} x_i x_j \\
&\quad + \sum_{i=1}^{k}\sum_{j=k+1}^{n} p^{i+j} x_i x_j + \sum_{i=k+1}^{n}\sum_{j=k+1}^{n} p^{i+j} x_i x_j - \sum_{i=1}^{n}\sum_{j=1}^{n} p^{i+j} x_i x_j \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} p^{i+j} y_{ij} - \sum_{i=1}^{k}\sum_{j=1}^{k} p^{i+j} x_i x_j \\
&\geq p^{2k} y_{kk} - (p^{2k}(x_k)^2 + p^{2k-1}(k^2-1)) \\
&= p^{2k}(x_k - (x_k)^2) - p^{2k-1}(k^2-1) \\
&> p^2(x_k - (x_k)^2) - pn^2 \geq p^2\varepsilon/2 - pn^2.
\end{aligned}
$$

The above theorem says that when p is sufficiently large, the vector $\boldsymbol{x} \in [0,1]^n$ is 0–1 valued if and only if $\sum_{i=1}^{n}\sum_{j=1}^{n} p^{i+j} y_{ij} - ([\boldsymbol{x}]_p)^2$ is non-positive. This result gives an idea to show NP-hardness of Problem P4. To show NP-hardness of Problem P4, we have to transform an NP-complete problem to the decision version of P4. Here we use the following NP-complete problem.

SET PARTITION [2, 3]

INSTANCE : An $m \times n$ 0–1 matrix M satisfying $n > m$.

QUESTION : Is there a 0–1 vector $\boldsymbol{x}$ satisfying $M\boldsymbol{x} = \mathbf{1}$? (Here, $\mathbf{1}$ denotes the all one vector.)

Then, it is natural to consider the following problem:

$$
\begin{aligned}
(\text{P4}(M))\quad & \text{minimize} \ \ \textstyle\sum_{i=1}^{n}\sum_{j=1}^{n} p^{i+j} y_{ij} - (\sum_{i=1}^{n} p^i x_i)^2 \\
& \text{subject to (3) and } M\boldsymbol{x} = \mathbf{1},
\end{aligned}
$$

where M is an $m \times n$ 0–1 matrix with $n > m$. Clearly, when the equality system $M\boldsymbol{x} = \mathbf{1}$ has a 0–1 valued solution, the optimal value of the above problem is less than or equal to zero. We will discuss the case that $M\boldsymbol{x} = \mathbf{1}$ does not have any 0–1 valued solution. The feasible region of Problem P4(M), denoted by $\Omega(M)$, is a bounded polytope. The number of constraints of Problem P4(M) is equal to $n + n^2 + 4(n^2 - n) + n + m$ and so the number of constraints is less than n^3, when $n \geq 5$. Let $(\boldsymbol{x}', \boldsymbol{y}')$ be a vertex of the polytope $\Omega(M)$. Since each coefficient of constraints is -1, 0 or 1, Cramer's rule implies that each element of $(\boldsymbol{x}', \boldsymbol{y}')$ is

0–1 valued or contained in the interval $[1/(n^3)^{n^3}, 1 - 1/(n^3)^{n^3}]$. This observation implies the following property.

THEOREM 2.2. *Let M be an $m \times n$ 0–1 matrix with $n > m$ and $n \geq 5$. Assume that $p = n^{n^4}$. The equality system $M\boldsymbol{x} = \mathbf{1}$ has a 0–1 valued solution if and only if the optimal value of Problem P4(M) is non-positive. When $M\boldsymbol{x} = \mathbf{1}$ does not have any 0–1 valued solution, the optimal value of P4(M) is greater than p.*

Proof. If $M\boldsymbol{x} = \mathbf{1}$ has a 0–1 valued solution, it is clear that the optimal value of Problem P4(M) is non-positive.

We consider the case that $M\boldsymbol{x} = \mathbf{1}$ does not have any 0–1 valued solution. For any vertex $(\boldsymbol{x}', \boldsymbol{y}')$ of the polytope $\Omega(M)$, each element of $(\boldsymbol{x}', \boldsymbol{y}')$ is 0–1 valued or contained in the interval $[1/(n^3)^{n^3}, 1 - 1/(n^3)^{n^3}]$. Since $M\boldsymbol{x} = \mathbf{1}$ does not have any 0–1 valued solution, $\boldsymbol{x}'$ is not 0–1 valued. Lemma 2.1 implies that:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} p^{i+j} y'_{ij} - (\sum_{i=1}^{n} p^i x'_i)^2 > p^2/(2(n^3)^{n^3}) - pn^2 = p(n^{n^4}/(2n^{3n^3}) - n^2) > p.$$

For any feasible solution $(\boldsymbol{x}, \boldsymbol{y})$ of P4(M), $(\boldsymbol{x}, \boldsymbol{y})$ is represented by a convex combination of vertices of $\Omega(M)$. Since the objective function of P4(M) is concave, every feasible solution $(\boldsymbol{x}, \boldsymbol{y})$ satisfy the inequality $\sum_{i=1}^{n}\sum_{j=1}^{n} p^{i+j} y_{ij} - (\sum_{i=1}^{n} p^i x_i)^2 > p$. ■

From the above, we can decide the answer to SET PARTITION by solving Problem P4(M). The input size of the largest coefficient appearing in P4(M) is $\lceil \log(p^{2n}) \rceil + 1 = \lceil \log(n^{n^4})^{2n} \rceil + 1 = \lceil 2n^5 \log n \rceil + 1$, and so the input size of Problem P4(M) is bounded by a polynomial of n. It implies that Problem P4 is NP-hard.

We can extend the above result to a more general global optimization problem.

COROLLARY 2.3. *Let n be a positive integer with $n \geq 5$ and we use p for n^{n^4}. Assume that $g(x_0, y_0)$ is a function satisfying the conditions that:*

(1) $\forall x_0 \in [0, np^n]$, $\forall y_0 \in [0, n^2 p^{2n}]$, if $y_0 - x_0^2 \leq 0$ then $g(x_0, y_0) \leq 0$,

(2) $\forall x_0 \in [0, np^n]$, $\forall y_0 \in [0, n^2 p^{2n}]$, if $y_0 - x_0^2 > p$ then $g(x_0, y_0) > 0$.

Given an $m \times n$ 0–1 matrix M with $n > m$ and $n \geq 5$, we define the problem:

$$\begin{array}{lll} (Pg(M)) & \textit{minimize} & g(x_0, y_0) \\ & \textit{subject to} & (3) \textit{ and } M\boldsymbol{x} = \mathbf{1}, \\ & & x_0 = \sum_{i=1}^{n} p^i x_i, \\ & & y_0 = \sum_{i=1}^{n}\sum_{j=1}^{n} p^{i+j} y_{ij}. \end{array}$$

Then, the optimal value of Pg(M) is non-positive if and only if the equality system $M\boldsymbol{x} = \mathbf{1}$ has a 0–1 valued solution.

3. Main Results

First, we show NP-hardness of Problem P1. We consider the special function:

$$\begin{aligned} g_1(x_0, y_0) &= (y_0 - p + 2p^{4n})^2 - 4p^{4n}x_0^2 - 4p^{8n} \\ &= (y_0 - p + 2p^{4n} + 2p^{2n}x_0)(y_0 - p + 2p^{4n} - 2p^{2n}x_0) - 4p^{8n} \end{aligned}$$

where $p = n^{n^4}$ and $n \geq 5$. We can show that $g_1(x_0, y_0)$ satisfies the conditions in Corollary 2.3 as follows.

(1) If (x_0, y_0) satisfies $x_0 \in [0, np^n]$, $y_0 \geq 0$ and $y_0 - x_0^2 \leq 0$, then

$$\begin{aligned} g_1(x_0, y_0) &\leq (y_0 - p)^2 + 2(y_0 - p)2p^{4n} + 4p^{8n} - 4p^{4n}y_0 - 4p^{8n} \\ &\leq (y_0 - p)^2 - 4p^{4n+1} \leq (y_0)^2 + p^2 - 4p^{4n+1} \\ &\leq (x_0)^4 + p^2 - 4p^{4n+1} \leq n^4p^{4n} + p^2 - 4p^{4n+1} \leq 0. \end{aligned}$$

(2) If (x_0, y_0) satisfies $y_0 - x_0^2 > p$ and $y_0 \geq 0$, then

$$g_1(x_0, y_0) > (x_0^2 + 2p^{4n})^2 - 4p^{4n}x_0^2 - 4p^{8n} = x_0^4 \geq 0.$$

From the above, we can show NP-hardness of the problem:

$$\begin{aligned} (\text{P1}(M)) \quad & \text{minimize} \quad z_1 z_2 \\ & \text{subject to} \quad (3) \text{ and } M\boldsymbol{x} = \mathbf{1}, \\ & \qquad x_0 = \textstyle\sum_{i=1}^{n} p^i x_i, \\ & \qquad y_0 = \textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} p^{i+j} y_{ij}, \\ & \qquad z_1 = (y_0 - p + 2p^{4n} + 2p^{2n}x_0), \\ & \qquad z_2 = (y_0 - p + 2p^{4n} - 2p^{2n}x_0). \end{aligned}$$

Corollary 2.3 implies that the optimal value of Problem P1(M) is less than or equal to $4p^{8n}$ if and only if $M\boldsymbol{x} = \mathbf{1}$ has a 0–1 valued solution. So, we have shown the following theorem.

THEOREM 3.1. *Problem P1 is NP-hard.*

Proof. When we solve Problem P1(M), we can decide the answer to SET PARTITION. The largest coefficient appearing in P1(M) is $2p^{4n} = 2(n^{n^4})^{4n} = 2n^{4n^5}$ and the threshold value is $4p^{8n} = 4(n^{n^4})^{8n} = 4n^{8n^5}$. Thus, the input size of Problem P1(M) and the input size of the threshold value are bounded by a polynomial of n. Clearly, Problem P1(M) is a special case of P1 and so we have the desired result. ■

Here we note that for any feasible solution of P1(M), both $z_1 > 0$ and $z_2 > 0$ hold. Since p is large enough, $z_1 > 0$ is clear. For the variable z_2,

$$z_2 \geq -p + 2p^{4n} - 2p^{2n}np^n = -p + 2p^{4n} - 2np^{3n}$$

and assumptions $n \geq 5$ and $p = n^{n^4}$ imply the property $z_2 > 0$.

Next, we consider Problem P2. Given three positive values z_1, z_2 and a, $z_1 z_2 \leq a^2$ if and only if $z_1 - a^2/z_2 \leq 0$. So, we decide the answer to SET PARTITION by solving the problem:

$$\begin{array}{lll} (\mathrm{P2}(M)) & \text{minimize} & z_1 - 1/z_3 \\ & \text{subject to} & \text{constraints of Problem P1}(M), \\ & & z_3 = z_2/(4p^{8n}). \end{array}$$

It is clear that the optimal value of P2(M) is non-positive if and only if the equality system $M\boldsymbol{x} = \mathbb{1}$ has a 0–1 valued solution. So, we have shown the following theorem.

THEOREM 3.2. *Problem P2 is NP-hard.*

Lastly, we consider Problem P3. Given three positive values z_1, z_2 and a, $z_1 z_2 \leq a^2$ if and only if $1/(z_1 + a) + 1/(z_2 + a) \geq 1/a$. Thus, we can decide the answer to SET PARTITION by solving the problem:

$$\begin{array}{lll} (\mathrm{P3}(M)) & \text{maximize} & 1/z_4 + 1/z_5 \\ & \text{subject to} & \text{constraints of Problem P1}(M), \\ & & z_4 = z_1 + 2p^{4n}, \\ & & z_5 = z_2 + 2p^{4n}. \end{array}$$

Clearly, the optimal value of P3(M) is greater than or equal to $1/2p^{4n}$ if and only if the equality system $M\boldsymbol{x} = \mathbb{1}$ has a 0-1 valued solution. So, we obtained the following.

THEOREM 3.3. *Problem P3 is NP-hard.*

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